SCALING SEPARABILITY CRITERION: APPLICATION TO GAUSSIAN STATES

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Abstract

We introduce examples of three- and four-mode entangled Gaussian mixed states that are not detected by the scaling and PeresHorodecki separability criteria. The presented modification of the scaling criterion resolves this problem. Also it is shown that the new criterion reproduces the main features of the scaling pictures for different cases of entangled states, while the previous versions lead to completely different outcomes. This property of the presented scheme is evidence of its higher generality.

Keywords: entanglement, Gaussian states, scaling transform, multimode light, separability criterion.

1. Introduction

Being one of the most striking phenomena in quantum physics, entanglement [1–3] has been thoroughly investigated for many years. One of the main research directions aims at finding the universal separability criterion [4–9]. The particular case of Gaussian states was investigated using the Peres-Horodecki separability criterion [4, 5], which is applicable to two-mode states that fail to work in general when applied to cases of higher dimensions. The PeresHorodecki criterion is based on the so-called ppt-transform (partial positive transpose transform), i.e., transpose of the density matrix of one part of a multipartite state, leaving other states untouched. If after the ppt-transform the density matrix of the whole system becomes physically meaningless, then the state is entangled. This is the essence of the PeresHorodecki separability criterion.

Our investigation concerns the previously developed extension of this criterion called "scaling separability criterion" [10–12]. In considering the density matrix transpose as a time reversal, we can think of generalizing this operation to time scaling. Doing so, the scaling criterion detects more Gaussian entangled states, yielding a nonphysical density matrix for at least one scaling parameter.

The separability criteria are closely connected with the question of finding the measure of entanglement (i.e., [13,14]), some of which are based on the PeresHorodecki criterion. The scaling criterion provides an intuitive visual measure as well, being not operational though. The criterion itself finds application in different cases discussed in [15–17].

In this paper, we provide a modification of the scaling criterion, which allows one to detect entanglement in a wide range of Gaussian pure and mixed states. Based on the results of our previous paper [12], in this work we investigate the use of the scaling criterion of separability, looking at cases for which the previous version of the criterion did not work as well as the PeresHorodecki criterion. Demonstrating the power of this method, we also discuss its higher generality, comparing the results for different related cases of mixed Gaussian states.

The paper is organized as follows.

In Sec. 2, we provide the basic theoretical background on the scaling criterion of separability applied to Gaussian states. Section 3 is devoted to the discussion of multimode uncertainty relations, which are needed for detecting nonphysical density matrices, on which the criterion is based. Finally, in Sec. 4 we provide examples of the method of operation for some three- and four-mode Gaussian mixed states, also discussing the aspects of the criterion, such as a comparison with its previous version and the PeresHorodecki criterion, and possible interpretation of the results, such as the entanglement measure, etc.

2. Scaling Criterion

Let us consider a single mode photon state with the density matrix ρ which should obey the following conditions:

$$\rho^{+} = \rho, \quad \mathrm{Tr}\rho = 1, \quad \rho \geq 0. \tag{1}$$

Also let $\hat{q}$ and $\hat{p}$ be the quadrature operators of this state. Then we can rewrite relations (1) in the form of Robertson-Schrödinger uncertainty relation [18,19]:

$$\begin{pmatrix} \sigma_{qq} & \sigma_{qp} - i/2 \\ \sigma_{qp} + i/2 & \sigma_{pp} \end{pmatrix} \geq 0 \tag{2}$$

where $\sigma_{\xi\zeta} = \left\langle \hat{\xi}\hat{\zeta} \right\rangle$ and the inequality is considered (here and further) in the sense of positivity of all principal minors of the matrix. This condition can be easily simplified

to

$$\Delta = \sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 \geq \frac{1}{4} \tag{3}$$

Now we'll transform the given state by multiplying its momentum by a scaling parameter λ. This is equivalent to the transform of time $t \to \lambda t$. Rewriting relation (3) for the modified state, we obtain the following condition, which should hold if the new state is physically realizable:

$$\frac{\Delta}{\lambda^2} = \frac{1}{\lambda^2}(\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2) \geq \frac{1}{4} \tag{4}$$

Hence, the uncertainty relation holds for $\lambda \in \left[-2\sqrt{\Delta}, 2\sqrt{\Delta}\right]$. It is worth mentioning that the transform performed in the Peres-Horodecki criterion ($t \to -t$ or $\rho \to \rho^T$) is included in this set of maps being represented by $\lambda = -1$ since Δ is always greater than $\frac{1}{4}$. The scaling criterion of separability considered in [12] used this transform for only $\lambda \in [-1, 1]$, which, as it will be shown later, is not enough for the entanglement detection.

The scaling criterion of separability for n-mode photon Gaussian states tells us that if we'll apply the scaling transform with coefficients $\lambda_i \in [-1, 1]$ to the momenta p_i of all the submodes, the modified state will become not physically realizable (the Robertson-Schrödinger uncertainty relations will not hold) for some set $\{\lambda_i\}$ only if the state is entangled. In [12] we proved that for pure three- and four- mode Gaussian states we can say "only if" and also that this criterion is more powerful than the Peres-Horodecki criterion in general. But there are some mixed entangled states that are not detected when using the scaling in the $[-1, 1]$ range. And further we will present some examples to show that these states can be detected by the criterion if we choose λ_i from the range $\left[-2\sqrt{\Delta_i}, 2\sqrt{\Delta_i}\right]$, where Δ_i is the value of the left-hand side of the uncertainty relation (3) for the ith submode.

3. Multimode Uncertainty Relations

There are many ways of checking if the state represented by some formula has a physical meaning. In our case, the simplest way is to check the fulfilment of the uncertainty relations in the RobertsonSchrödinger form.

Let us introduce these relations in the multimode case, their modification under scaling transform, and find the operations needed to apply them to the separability criterion.

The Wigner function of the generic Gaussian form for n-mode state reads

$$W(q, p) = \frac{1}{\sqrt{\det \sigma}} \exp\left(-\frac{1}{2}\mathbf{Q}\sigma^{-1}\mathbf{Q^T}\right), \tag{5}$$

where the $2n$-dimensional vector $\mathbf{Q}$ is

$$\mathbf{Q} = \left(q_1 - \langle q_1 \rangle, q_2 - \langle q_2 \rangle, ..., q_n - \langle q_n \rangle, p_1 - \langle p_1 \rangle, p_2 - \langle p_2 \rangle, ..., p_n - \langle p_n \rangle\right), \quad (6)$$

and the matrix σ is a $2n \times 2n$ real symmetric variance matrix

$$\sigma_{r_i r_j} = \frac{1}{2} \langle \hat{r}_i \hat{r}_j + \hat{r}_j \hat{r}_i \rangle, \quad (7)$$

where $\hat{r}_i = \hat{q}_i$, $\hat{r}_{n+j} = \hat{p}_j$, and $i, j = 1, ..., n$.

The matrix σ is a good characteristic of a Gaussian state by which one can judge on its reality using the uncertainty relation

$$\sigma + \frac{i}{2}\Omega \geq 0 \quad (8)$$

where Ω presented in a block form is

$$\Omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

and I is the identity matrix. Again, the inequality here is considered as a nonnegativity of the principal minors of the matrix.

It is easy to see that the scaling of the ith mode $p_i \to \lambda_i p_i$ is identical to the division of $(n+i)$th row and column of σ by λ_i. Since the first n principal minors remain unchanged, the check of the uncertainty relations should be performed for only minors of higher dimensions. Obviously, one should perform the scaling in the range identified above and check the positivity of each of the minors for all possible sets $\{\lambda_i\}$. But further we will be considering only the determinant; to the best of our knowledge, there are no entangled states that are detected by the lower minors and not detected by the determinant, and no entangled Gaussian states that are not detected by the presented algorithm.

4. Performance on Three- and Four-Mode Mixed States

Let us provide some examples for which the scaling criterion (and the Peres-Horodecki criterion) does not work when using only scaling from -1 to 1.

Performing the scaling of the three-mode mixed state with the dispersion matrix

$$\sigma = \begin{pmatrix} 6/5 & 1/5 & 1/5 & 1/10 & 1/10 & 1/10 \\ 1/5 & 6/5 & 1/5 & 1/10 & 1/10 & 1/10 \\ 1/5 & 1/5 & 6/5 & 1/10 & 1/10 & 1/10 \\ 1/10 & 1/10 & 1/10 & 1 & -1/8 & -1/8 \\ 1/10 & 1/10 & 1/10 & -1/8 & 1 & -1/8 \\ 1/10 & 1/10 & 1/10 & -1/8 & -1/8 & 1 \end{pmatrix} \quad (9)$$

we'll get some area of negativity within our scaling range although there is no negative results for $\lambda_i \in [-1, 1]$. This result is vividly depicted in Fig. 1 where the plot for $\det\left(\sigma + \frac{i}{2}\Omega\right)$ versus λ_2 and λ_3 is shown. Here $\lambda_1 = 1/2$, but other possible values of this coefficient will yield a very similar plot, also positive on $[1, 1]$. The grayscale depicts the level of negativity of the determinant (level of nonrealizability of the state how large is the violation of the uncertainty relations), and the white area stands for positive values that we are not interested in. For comparison, Fig. 2 shows the plot for σ with 1/5 in the upper-right and lower-left 4×4 quadrants. This is the one considered in our previous paper [12], and it is obvious that there are negative points within $\lambda_2 \times \lambda_3 \in [1, 1]$, while the first state is not detected as entangled either by the previous version of the scaling criterion or by the PeresHorodecki criterion.

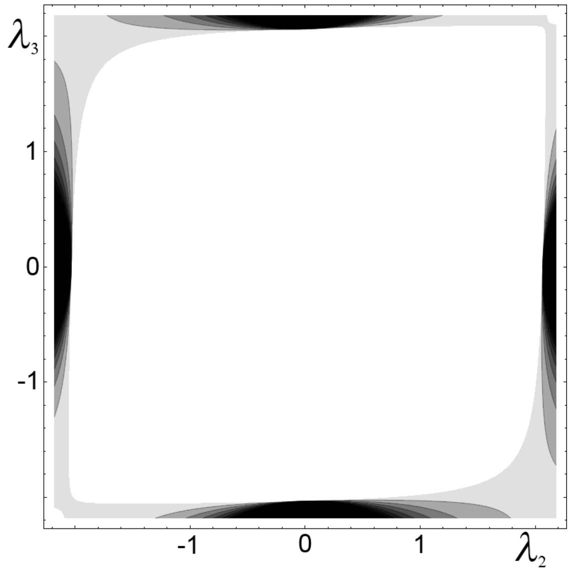

Fig. 1. The separability test plot for the three-mode state (9).

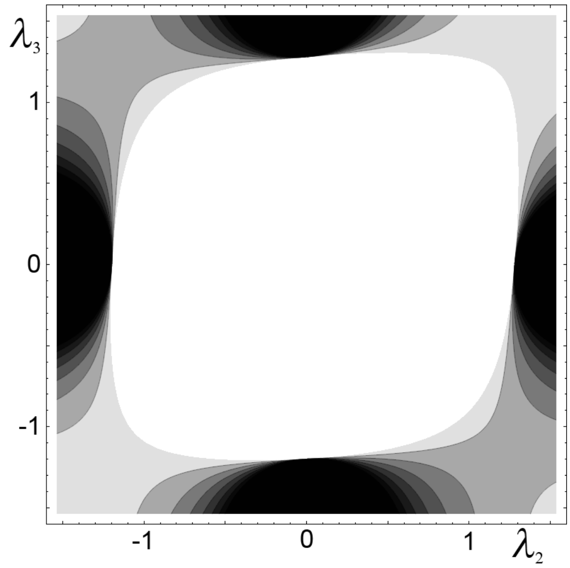

Fig. 2. The separability test plot for the modified three-mode state obtained from (9) by replacement of the matrix elements.

The same result is obtained for the following four-mode dispersion matrix

$$\sigma = \begin{pmatrix} 8/5 & 2/5 & 2/5 & 2/5 & 1/50 & 1/50 & 1/50 & 1/50 \\ 2/5 & 8/5 & 2/5 & 2/5 & 1/50 & 1/50 & 1/50 & 1/50 \\ 2/5 & 2/5 & 8/5 & 2/5 & 1/50 & 1/50 & 1/50 & 1/50 \\ 2/5 & 2/5 & 2/5 & 8/5 & 1/50 & 1/50 & 1/50 & 1/50 \\ 1/50 & 1/50 & 1/50 & 1/50 & 1 & -1/8 & -1/8 & -1/8 \\ 1/50 & 1/50 & 1/50 & 1/50 & -1/8 & 1 & -1/8 & -1/8 \\ 1/50 & 1/50 & 1/50 & 1/50 & -1/8 & -1/8 & 1 & -1/8 \\ 1/50 & 1/50 & 1/50 & 1/50 & -1/8 & -1/8 & -1/8 & 1 \end{pmatrix} \tag{10}$$

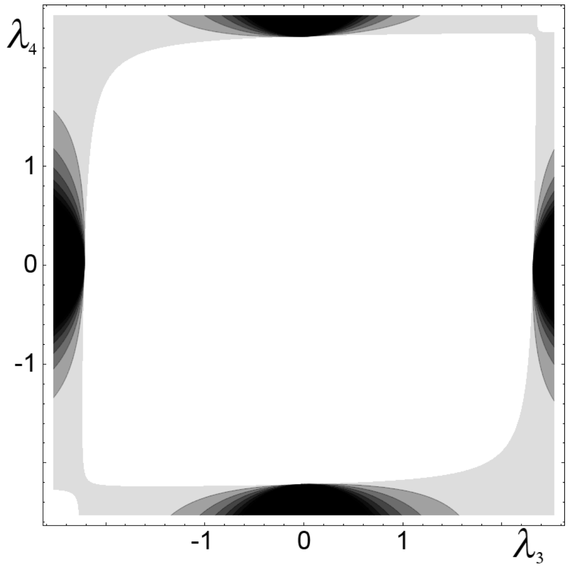

Fig. 3. The separability test plot for the four-mode state (10).

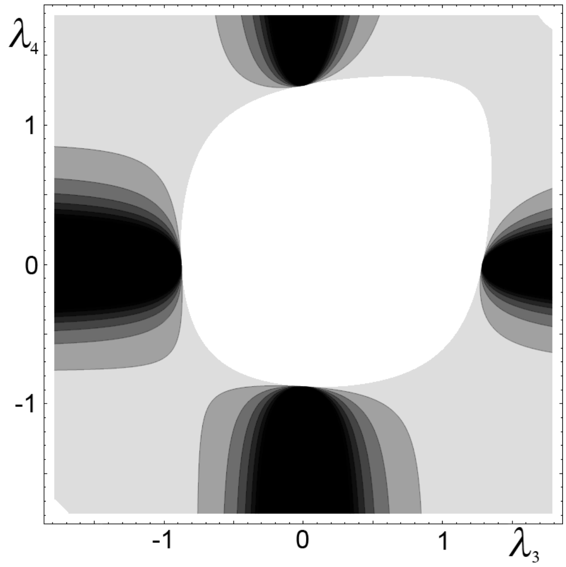

Fig. 4. The separability test plot for the modified four-mode state obtained from (10) by replacement of the matrix elements.

The plot for $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$ is shown in Fig. 3 and also the previously considered state [12] when the upper-right and lower-left 6×6 quadrants are filled with 1/10 as shown on the Fig. 4. Again, the first state turns out to be realizable for all $\lambda_i \in [1, 1]$, and so the entanglement would not be detected.

Now we see that this modified criterion reproduces the appearance of the scaling picture for different states (for more examples, refer to [12]) and detects the entanglement for a larger class of mixed states surpassing the power of its predecessors.

5. Conclusions

In this work, we showed that the new version of the scaling separability criterion is more powerful than the initial one which, in turn, is stronger than the PeresHorodecki criterion. Investigating some examples of three- and four-mode mixed Gaussian states, we also provided an intuitive argument in favor of the generality of this method, relating to the measure of entanglement as well.

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